

New Extension of Unified Family of Apostol-Type of Polynomials and Numbers

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Abstract

The purpose of this paper is to introduce and investigate a new unification of unified family of Apostol-type polynomials and numbers based on results given in [24] and [25]. Also, we derive some properties for these polynomials and obtain some relationships between the Jacobi polynomials, Laguerre polynomials, Hermite polynomials, Stirling numbers and some other types of generalized polynomials.

Key words: Generalized Euler, Bernoulli and Genocchi polynomials; Stirling numbers; generalized Stirling numbers; Laguerre polynomials; Hermite polynomials; Jacobi polynomials.

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1. Introduction

The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$ and the generalized Euler polynomials are defined by (see [23]):

$$\left(\frac{t}{e^t - 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}, \quad (|t| < 2\pi; 1^\alpha := 1) \quad (1.1)$$

and

$$\left(\frac{t}{e^t + 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!}, \quad (|t| < \pi; 1^\alpha := 1), \quad (1.2)$$

where \mathbb{C} denote set of complex numbers.

Recently, Luo and Srivastava [14] introduced the generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x; \lambda)$ and the generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x; \lambda)$ as follows:

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Definition 1.1. (Luo and Srivastava [14]) The generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x; \lambda)$ of order $\alpha \in \mathbb{C}$ are defined by the generating function

$$\left(\frac{t}{\lambda e^t - 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}$$

$$(|t| < 2\pi \text{ when } \lambda = 1; |t| < |\log \lambda|, \text{ when } \lambda \neq 1; 1^\alpha := 1).$$
 (1.3)

Definition 1.2. (Luo [15]) The generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x; \lambda)$ of order $\alpha \in \mathbb{C}$ are defined by the generating function

$$\left(\frac{t}{\lambda e^t + 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}$$

$$(|t| < \pi \text{ when } \lambda = 1; |t| < |\log(-\lambda)|, \text{ when } \lambda \neq 1; 1^\alpha := 1).$$
 (1.4)

Natalini and Bernardini [17] defined the new generalization of Bernoulli polynomials in the following form.

Definition 1.3. The generalized Bernoulli polynomials $B_n^{[m-1]}(x)$, $m \in \mathbb{N}$, are defined, in a suitable neighbourhood of $t = 0$ by means of generating function

$$\frac{t^m e^{xt}}{e^t - \sum_{l=0}^{m-1} \frac{t^l}{l!}} = \sum_{n=0}^{\infty} B_n^{[m-1]}(x) \frac{t^n}{n!}. \quad (1.5)$$

Recently, Tremblay et al. [26] investigated a new class of generalized Apostol-Bernoulli polynomials. These are defined as follows.

Definition 1.4. The generalized Apostol-Bernoulli polynomials $B_n^{[m-1, \alpha]}(x; \lambda)$ of order $\alpha \in \mathbb{C}$, $m \in \mathbb{N}$, are defined, in a suitable neighbourhood of $t = 0$ by means of generating function

$$\left(\frac{t^m}{\lambda e^t - \sum_{l=0}^{m-1} \frac{t^l}{l!}}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{[m-1, \alpha]}(x; \lambda) \frac{t^n}{n!}. \quad (1.6)$$

Also, Srivastava et al. [24] introduced a new interesting class of Apostol-Bernoulli polynomials that are closely related to the new class that we present in this paper. They investigated the following form.

Definition 1.5. Let $a, b, c \in \mathbb{R}^+$ ($a \neq b$) and $n \in \mathbb{N}_0$. Then the generalized Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x; \lambda; a, b, c)$ of order $\alpha \in \mathbb{C}$ are defined by the following generating function:

$$\left(\frac{t}{\lambda b^t - a^t}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(\alpha)}(x; \lambda; a, b, c) \frac{t^n}{n!}$$

$$\left(\left|t \log\left(\frac{a}{b}\right)\right| < |\log \lambda|; 1^\alpha := 1\right). \quad (1.7)$$

In this sequel to the work by Sirvastava et al. [25] introduced and investigated a similar generalization of the family of Euler polynomials defined as follows.

Definition 1.6. Let $a, b, c \in \mathbb{R}^+(a \neq b)$ and $n \in \mathbb{N}_0$. Then the generalized Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x; \lambda; a, b, c)$ of order $\alpha \in \mathbb{C}$ are defined by the following generating function:

$$\left(\frac{t}{\lambda b^t + a^t} \right)^\alpha c^{xt} = \sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x; \lambda; a, b, c) \frac{t^n}{n!}$$

$$\left(\left| t \log \left(\frac{a}{b} \right) \right| < |\log(-\lambda)|; 1^\alpha := 1 \right). \quad (1.8)$$

It is easy to see that setting $a = 1$ and $b = c = e$ in (1.8) would lead to Apostol-Euler polynomials defined by (1.4). The case where $\alpha = 1$ has been studied by Luo et al. [12].

In Section 2, we introduce the new extension of unified family of Apostol-type polynomials and numbers that are defined in [7]. Also, we determine relation between some results given in [23, 24, 10, 11, 26] and our results and introduce some new identities for polynomials defined in [7]. In Section 3, we give some basic properties of the new unification of Apostol-type polynomials and numbers. Finally in Section 4, we introduce some relationships between the new unification of Apostol-type polynomials and other known polynomials.

2. Unification of multiparameter Apostol-type polynomials and numbers

Definition 2.1. Let $a, b, c \in \mathbb{R}^+(a \neq b)$, $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Then the new unification of Apostol-type polynomials $M_n^{[m-1, r]}(x; k; a, b, c; \bar{\alpha}_r)$ are defined, in a suitable neighbourhood of $t = 0$ by means of generating function

$$F_{\bar{\alpha}_r}^{[m-1, r]} = \frac{t^{rkm} 2^{rm(1-k)} c^{xt}}{\prod_{i=0}^{r-1} \left(\alpha_i b^t - a^t \sum_{\ell=0}^{m-1} \frac{t^\ell}{\ell!} \right)} = \sum_{n=0}^{\infty} M_n^{[m-1, r]}(x; k; a, b, c; \bar{\alpha}_r) \frac{t^n}{n!}$$

$$\left(\left| t \log \left(\frac{b}{a} \right) \right| < 2\pi \text{ when } m = 1 \text{ and } \alpha_i = 1; \left| t \log \left(\frac{b}{a} \right) \right| < |\log(\alpha_i)| \right.$$

$$\left. \text{when } m = 1 \text{ and } \alpha_i \neq 1; \forall i = 0, 1, \dots, r-1 \right), \quad (2.1)$$

where $k \in \mathbb{N}_0; r \in \mathbb{C}; \bar{\alpha}_r = (\alpha_0, \alpha_1, \dots, \alpha_{r-1})$ is a sequence of complex numbers.

The generating function in (2.1) gives many types of polynomials as special cases, for example, see the following table

Table 1:

1	setting $k = 1, \alpha_i = \lambda, i = 0, 1, \dots, r - 1$, hence if $m = 1$ in (2.1)	$M_n^{[0,r]}(x; 1; a, b, c; \lambda) = \mathfrak{B}_n^{(r)}(x; \lambda; a, b, c)$ (generalized Bernoulli polynomials of order r , see [25])
2	setting $k = 0, \alpha_i = -\lambda, i = 0, 1, \dots, r - 1$, hence if $m = 1$ in (2.1)	$M_n^{[0,r]}(x; 0; a, b, c; -\lambda) = (-1)^r \mathfrak{E}_n^{(r)}(x; \lambda; a, b, c)$ (generalized Euler polynomials of order r , see [25])
3	setting $\alpha_i = \beta, i = 0, 1, \dots, r - 1, c = b$, hence if $m = 1$ in (2.1)	$M_n^{[0,r]}(x; k; a, b, b; \beta) = y_{n,\beta}^{(r)}(x; k; a, b)$ (unification of Apostol-type polynomials of order r , see [21])
4	setting $k = 1, t = t \ln a, x = \frac{x}{\ln a}, \alpha_i = \lambda, i = 0, 1, \dots, r - 1$, hence if $a = 1, b = c^{\frac{1}{\ln a}}$ in (2.1)	$M_n^{[m-1,r]}(\frac{x}{\ln a}; 1, 1, c^{\frac{1}{\ln a}}, c; \lambda) = (\ln a)^{mr} B_n^{[m-1,r]}(x; c, a; \lambda)$ (generalized Bernoulli polynomials of order r , see [11])
5	setting $k = 0, t = t \ln a, x = \frac{x}{\ln a}, \alpha_i = -\lambda, i = 0, 1, \dots, r - 1$, hence if $a = 1, b = c^{\frac{1}{\ln a}}$ in (2.1)	$M_n^{[m-1,r]}(\frac{x}{\ln a}; 0; 1, c^{\frac{1}{\ln a}}, c; -\lambda) = (-1)^r (\ln a)^{mr} E_n^{[m-1,r]}(x; c, a; \lambda)$ (generalized Euler polynomials of order r , see [11])
6	setting $k = 1, \alpha_i = 1, i = 0, 1, \dots, r - 1, a = 1, b = e, c = e$, hence if $r = 1$ in (2.1)	$M_n^{[m-1,1]}(x; 1; 1, e, e, 1) = B_n^{[m-1]}(x)$ (generalized Bernoulli polynomials, see [17])
7	setting $k = 0, \alpha_i = -1, i = 0, 1, \dots, r - 1, a = 1, b = e, c = e$, hence if $r = 1$ in (2.1)	$M_n^{[m-1,1]}(x; 0; 1, e, e, -1) = -E_n^{[m-1]}(x)$ (generalized Euler polynomials, see [17])
8	setting $k = 1, \alpha_i = 1, i = 0, 1, \dots, r - 1, a = 1, b = e, c = e$ in (2.1)	$M_n^{[m-1,r]}(x; 1; 1, e, e, 1) = B_n^{[m-1,r]}(x)$ (generalized Bernoulli polynomials of order r , see [10])
9	setting $k = 0, \alpha_i = -1, i = 0, 1, \dots, r - 1, a = 1, b = e, c = e$ in (2.1)	$M_n^{[m-1,r]}(x; 0; 1, e, e, -1) = (-1)^r E_n^{[m-1,r]}(x)$ (generalized Euler polynomials of order r , see [10])
10	setting $k = 1, \alpha_i = -1, i = 0, 1, \dots, r - 1, a = 1, b = e, c = e$ in (2.1)	$M_n^{[m-1,r]}(x; 1; 1, e, e, -1) = (-1)^r \left(\frac{1}{2}\right)^{rm} G_n^{[m-1,r]}(x)$ (generalized Genocchi polynomials of order r , see [10])
11	setting $k = 1, \alpha_i = \lambda, i = 0, 1, \dots, r - 1, a = 1, b = e, c = e$ in (2.1)	$M_n^{[m-1,r]}(x; 1; 1, e, e; \lambda) = B_n^{[m-1,r]}(x; \lambda)$ (generalized Apostol-Bernoulli polynomials of order r , see [26])
12	setting $k = 0, \alpha_i = -\lambda, i = 0, 1, \dots, r - 1, a = 1, b = e, c = e$ in (2.1)	$M_n^{[m-1,r]}(x; 0; 1, e, e; -\lambda) = (-1)^r E_n^{[m-1,r]}(x; \lambda)$ (generalized Apostol-Euler polynomials of order r , see [26])
13	setting $m = 1, a = 1, b = e, c = e$ in (2.1)	$M_n^{[0,r]}(x; k; 1, e, e; -\bar{\alpha}_r) = M_n^{(r)}(x; k; \bar{\alpha}_r)$ (a new unified family of generalized Apostol-Euler, Bernoulli and Genocchi polynomials, see [7])

Remark 2.1. If we set $x = 0$ in (2.1), then we obtain the new unification of multiparameter Apostol-type numbers, as

$$M_n^{[m-1,r]}(0; k; a, b, c; \overline{\alpha}_r) = M_n^{[m-1,r]}(k; a, b, c; \overline{\alpha}_r). \quad (2.2)$$

Remark 2.2. From **No.13** in **Table1** and [7, **Table1**], we can obtain the polynomials and the numbers given in [1, 5, 9, 13, 21].

3. Some basic properties for the polynomial $M_n^{[m-1,r]}(x; k; a, b, c; \overline{\alpha}_r)$

Theorem 3.1. Let $a, b, c \in \mathbb{R}^+(a \neq b)$ and $x \in \mathbb{R}$. Then

$$M_n^{[m-1,r]}(x+y; k; a, b, c; \overline{\alpha}_r) = \sum_{l=0}^n \binom{n}{l} x^{n-l} (\ln c)^{n-l} M_l^{[m-1,r]}(y; k; a, b, c; \overline{\alpha}_r). \quad (3.1)$$

$$M_n^{[m-1,r]}(x+r; k; a, b, c; \overline{\alpha}_r) = M_n^{[m-1,r]} \left(x; k; \frac{a}{c}, \frac{b}{c}, c; \overline{\alpha}_r \right). \quad (3.2)$$

Proof. For the first equation, from (2.1)

$$\begin{aligned} \sum_{n=0}^{\infty} M_n^{[m-1,r]}(x+y; k; a, b, c; \overline{\alpha}_r) \frac{t^n}{n!} &= \frac{t^{rkm} 2^{rm(1-k)} c^{xt}}{\prod_{i=0}^{r-1} \left(\alpha_i b^t - a^t \sum_{\ell=0}^{m-1} \frac{t^\ell}{\ell!} \right)} c^{yt} \\ &= \sum_{j=0}^{\infty} \frac{(ty \ln c)^j}{j!} \sum_{l=0}^{\infty} M_l^{[m-1,r]}(x; k; a, b, c; \overline{\alpha}_r) \frac{t^l}{l!}, \end{aligned}$$

using Cauchy product rule, we can easily obtain (3.1).

For the second equation (3.2), from (2.1)

$$\begin{aligned} \sum_{n=0}^{\infty} M_n^{[m-1,r]}(x+r; k; a, b, c; \overline{\alpha}_r) \frac{t^n}{n!} &= \frac{t^{rkm} 2^{rm(1-k)}}{\prod_{i=0}^{r-1} \left(\alpha_i \left(\frac{b}{c} \right)^t - \left(\frac{a}{c} \right)^t \sum_{\ell=0}^{m-1} \frac{t^\ell}{\ell!} \right)} c^{xt} \\ &= \sum_{n=0}^{\infty} M_n^{[m-1,r]} \left(x; k; \frac{a}{c}, \frac{b}{c}, c; \overline{\alpha}_r \right) \frac{t^n}{n!}. \end{aligned}$$

Equating coefficient of $\frac{t^n}{n!}$ on both sides, yields (3.2). \square

Corollary 3.1. If $y = 0$ in (3.1), we have

$$M_n^{[m-1,r]}(x; k; a, b, c; \overline{\alpha}_r) = \sum_{\ell=0}^n \binom{n}{\ell} x^{n-\ell} (\ln c)^{n-\ell} M_\ell^{[m-1,r]}(k; a, b, c; \overline{\alpha}_r) \quad (3.3)$$

$$= \sum_{\ell=0}^n \binom{n}{n-\ell} x^\ell (\ln c)^\ell M_{n-\ell}^{[m-1,r]}(k; a, b, c; \overline{\alpha}_r). \quad (3.4)$$

Theorem 3.2. *The following identity holds true, when $m = 1$ and $\alpha_i \neq 0$ in (2.1) $\forall i = 0, 1, \dots, r-1$*

$$M_n^{[0,r]}(r-x; k; a, b, c; \bar{\alpha}_r) = \frac{(-1)^{r(1-k)+n}}{\prod_{i=0}^{r-1} \alpha_i} \sum_{m=0}^n \binom{n}{m} \left(r \ln \left(\frac{ab}{c} \right) \right)^{n-m} M_m^{[0,r]} \left(x; k; a, b, c; \frac{1}{\bar{\alpha}_r} \right). \quad (3.5)$$

Proof. From (2.1)

$$\begin{aligned} \sum_{n=0}^{\infty} M_n^{[0,r]}(r-x; k; a, b, c; \bar{\alpha}_r) \frac{t^n}{n!} &= \frac{t^{rk} 2^{r(1-k)} c^{(r-x)t}}{\prod_{i=0}^{r-1} (\alpha_i b^t - a^t)} \\ &= \frac{(-1)^{r(1-k)}}{(b^t a^t)^r \prod_{j=0}^{r-1} \alpha_j} \frac{(-t)^{rk} 2^{r(1-k)} c^{-xt}}{\prod_{i=0}^{r-1} \left(\frac{b^{-t}}{\alpha_i} - a^{-t} \right)} e^{rt} \\ &= \frac{(-1)^{r(1-k)}}{\prod_{j=0}^{r-1} \alpha_j} \left(\frac{ba}{c} \right)^{-rt} \sum_{m=0}^{\infty} M_m^{[0,r]} \left(x; k; a, b, c; \frac{1}{\bar{\alpha}_r} \right) \frac{(-t)^m}{m!} \\ &= \frac{(-1)^{r(1-k)}}{\prod_{j=0}^{r-1} \alpha_j} \sum_{\ell=0}^{\infty} \frac{(r \ln \left(\frac{ab}{c} \right))^{\ell}}{\ell!} (-t)^{\ell} \sum_{m=0}^{\infty} M_m^{[0,r]} \left(x; k; a, b, c; \frac{1}{\bar{\alpha}_r} \right) \frac{(-t)^m}{m!}. \end{aligned}$$

Hence, we can easily obtain (3.5). \square

Remark 3.1. *If we put $\alpha_i = \beta, i = 0, 1, \dots, r-1$, $c = b$ and $r = v$ in (3.5), then it gives [21, Eq. (34)],*

$$M_n^{[0,v]}(v-x; k; a, b, b; \beta) = \frac{(-1)^{v(1-k)+n}}{(\beta)^v} \sum_{m=0}^n \binom{n}{m} ((v \ln a)^{n-m} M_m^{[0,v]}(x; k; a, b, b; \beta^{-1})),$$

where $M_m^{[0,v]}(x; k; a, b, b; \beta^{-1})$ is the unification of the Apostol-type polynomials.

Theorem 3.3. *The unification of Apostol-type numbers satisfy*

$$M_n^{[m-1,r]}(k; a, b, c; \bar{\alpha}_r) = \sum_{l=0}^n \binom{n}{l} M_l^{[m-1,\ell]}(k; a, b, c; \bar{\alpha}_\ell) M_{n-l}^{[m-1,r-\ell]}(k; a, b, c; \bar{\alpha}_{r-\ell}). \quad (3.6)$$

Proof. When $x = 0$ in (2.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} M_n^{[m-1,r]}(k; a, b, c; \bar{\alpha}_r) \frac{t^n}{n!} &= \frac{t^{rk} 2^{r(1-k)}}{\prod_{i=0}^{r-1} \left(\alpha_i b^t - a^t \sum_{\ell=0}^{m-1} \frac{t^\ell}{\ell!} \right)} \\ &= \frac{t^{\ell k} 2^{\ell(1-k)}}{\prod_{i=0}^{\ell-1} \left(\alpha_i b^t - a^t \sum_{\ell=0}^{m-1} \frac{t^\ell}{\ell!} \right)} \frac{t^{(r-\ell)k} 2^{(r-\ell)(1-k)}}{\prod_{i=\ell}^{r-1} \left(\alpha_i b^t - a^t \sum_{\ell=0}^{m-1} \frac{t^\ell}{\ell!} \right)} \\ &= \sum_{\ell_1=0}^{\infty} M_{\ell_1}^{[m-1,\ell]}(k; a, b, c; \bar{\alpha}_\ell) \frac{t^{\ell_1}}{\ell_1!} \sum_{\ell_2=0}^{\infty} M_{\ell_2}^{[m-1,r-\ell]}(k; a, b, c; \bar{\alpha}_{r-\ell}) \frac{t^{\ell_2}}{\ell_2!}. \end{aligned}$$

Using Cauchy product rule, we obtain (3.6). \square

Theorem 3.4. *The following relationship holds true*

$$\sum_{k_1+k_2+\dots+k_\ell=n} \prod_{i=1}^{\ell} \frac{M_{k_i}^{[m-1, r_i]}(x_i; k; a, b, c; \bar{\alpha}_{r_i})}{k_1! k_2! \dots k_\ell!} = \frac{1}{n!} M_n^{[m-1, |\mathbf{r}|]}(|\mathbf{x}|; k; a, b, c; \bar{\alpha}_{|\mathbf{r}|}), \quad (3.7)$$

where $|\mathbf{r}| = r_1 + r_2 + \dots + r_\ell$ and $|\mathbf{x}| = x_1 + x_2 + \dots + x_\ell$ and $\bar{\alpha}_{r_i} = (\alpha_{\sum_{j=1}^{i-1} r_j}, \alpha_{\sum_{j=1}^{i-1} r_j + 1}, \dots, \alpha_{\sum_{j=1}^i r_j - 1})$, $i = \{1, 2, \dots, \ell\}$.

Proof. Starting with (2.1), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \left(M_n^{[m-1, |\mathbf{r}|]}(|x|; k; a, b, c; \bar{\alpha}_{|\mathbf{r}|}) \right) \frac{t^n}{n!} &= \frac{t^{|\mathbf{r}|} k^m 2^{|\mathbf{r}|m(1-k)} c^{|\mathbf{x}|t}}{\prod_{i=0}^{|\mathbf{r}|-1} \left(\alpha_i b^t - a^t \sum_{\ell=0}^{m-1} \frac{t^\ell}{\ell!} \right)} \\ &= \frac{t^{r_1 k m} 2^{r_1 m(1-k)} c^{x_1 t}}{\prod_{i=0}^{r_1-1} \left(\alpha_i b^t - a^t \sum_{\ell=0}^{m-1} \frac{t^\ell}{\ell!} \right)} \frac{t^{r_2 k m} 2^{r_2 m(1-k)} c^{x_2 t}}{\prod_{i=r_1}^{r_1+r_2-1} \left(\alpha_i b^t - a^t \sum_{\ell=0}^{m-1} \frac{t^\ell}{\ell!} \right)} \dots \frac{t^{r_\ell k m} 2^{r_\ell m(1-k)} c^{x_\ell t}}{\prod_{i=r_1+r_2+\dots+r_{\ell-1}}^{r_1+r_2+\dots+r_\ell-1} \left(\alpha_i b^t - a^t \sum_{\ell=0}^{m-1} \frac{t^\ell}{\ell!} \right)} \\ &= \sum_{k_1=0}^{\infty} M_{k_1}^{[m-1, r_1]}(x_1; k; a, b, c; \bar{\alpha}_{r_1}) \frac{t^{k_1}}{k_1!} \sum_{k_2=0}^{\infty} M_{k_2}^{[m-1, r_2]}(x_2; k; a, b, c; \bar{\alpha}_{r_2}) \frac{t^{k_2}}{k_2!} \dots \sum_{k_\ell=0}^{\infty} M_{k_\ell}^{[m-1, r_\ell]}(x_\ell; k; a, b, c; \bar{\alpha}_{r_\ell}) \frac{t^{k_\ell}}{k_\ell!} \end{aligned}$$

Using Cauchy product rule on the right hand side of the last equation and equating coefficients of t^n on both sides, yields (3.7). \square

Using **No.13 in Table1**, we obtain *Nörlund's* results, see [19] and *Carlitz's* generalizations, see [2] by our approach in Theorem 3.5 and Theorem 3.6 as follows

Theorem 3.5. *For $(\bar{\alpha}_r)^n = (\alpha_0^n, \alpha_1^n, \dots, \alpha_{r-1}^n)$, we have*

$$\begin{aligned} \prod_{i=1}^r \sum_{s_i=0}^{n-1} (\alpha_{i-1})^{s_i} M_\ell^{[0, r]} \left(x + \frac{\sum_{i=1}^r s_i}{n}; k; 1, e, e; (\bar{\alpha}_r)^n \right) &= n^{r k - \ell} M_\ell^{[0, r]}(nx; k; 1, e, e; \bar{\alpha}_r). \quad (3.8) \\ \prod_{i=1}^r \sum_{s_i=0}^{n-1} (\alpha_{i-1})^{s_i} M_{r+\ell}^{[0, r]} \left(x + \frac{\sum_{i=1}^r s_i}{n}; k; 1, e, e; (\bar{\alpha}_r)^n \right) &= n^{r(k-1)-\ell} \frac{(\ell+r)!}{\ell!} M_\ell^{[0, r]}(nx; k-1; 1, e, e; \bar{\alpha}_r). \quad (3.9) \end{aligned}$$

Proof. For the first equation and starting with (2.1), we get

$$\begin{aligned} \sum_{\ell=0}^{\infty} \frac{(nt)^\ell}{\ell!} \prod_{i=1}^r \sum_{s_i=0}^{n-1} (\alpha_{i-1})^{s_i} M_\ell^{[0, r]} \left(x + \frac{\sum_{i=1}^r s_i}{n}; k; 1, e, e; (\bar{\alpha}_r)^n \right) &= \frac{(nt)^{rk} 2^{r(1-k)} e^{nxt}}{\prod_{i=0}^{r-1} \alpha_i^n e^{nt} - 1} \prod_{i=1}^r \sum_{s_i=0}^{n-1} (\alpha_{i-1} e^t)^{s_i} \\ &= \frac{(nt)^{rk} 2^{r(1-k)} e^{(nx)t}}{\prod_{i=0}^{r-1} \alpha_i e^t - 1} = n^{rk} \sum_{\ell=0}^{\infty} M_\ell^{[0, r]}(nx; k; 1, e, e; \bar{\alpha}_r) \frac{t^\ell}{\ell!}. \end{aligned}$$

Equating coefficients of t^ℓ on both sides, yields (3.8).
For the second equation and starting with (2.1), we get

$$\begin{aligned} \sum_{\ell=0}^{\infty} \frac{(nt)^\ell}{\ell!} \prod_{i=1}^r \sum_{s_i=0}^{n-1} (\alpha_{i-1})^{s_i} M_\ell^{[0,r]} \left(x + \frac{\sum_{i=1}^r s_i}{n}; k; 1, e, e; (\bar{\alpha}_r)^n \right) &= \frac{n^{rk} t^r 2^{-r} (t)^{r(k-1)} 2^{r(2-k)} e^{nxt}}{\prod_{i=0}^{r-1} \alpha_i^n e^{nt} - 1} \prod_{i=1}^r \sum_{s_i=0}^{n-1} (\alpha_{i-1} e^t)^{s_i} \\ &= \frac{n^{rk} t^r 2^{-r} (t)^{r(k-1)} 2^{r(2-k)} e^{nxt}}{\prod_{i=0}^{r-1} \alpha_i e^t - 1}, \end{aligned}$$

then, we have

$$\sum_{\ell=0}^{\infty} \frac{n^{\ell+r} \ell!}{(\ell+r)!} \frac{t^\ell}{\ell!} \prod_{i=1}^r \sum_{s_i=0}^{n-1} (\alpha_{i-1})^{s_i} M_{r+\ell}^{[0,r]} \left(x + \frac{\sum_{i=1}^r s_i}{n}; k; 1, e, e; (\bar{\alpha}_r)^n \right) = n^{rk} 2^{-r} \sum_{\ell=0}^{\infty} M_\ell^{[0,r]}(nx; k-1; 1, e, e; \bar{\alpha}_r) \frac{t^\ell}{\ell!}.$$

Equating coefficients of t^ℓ on both sides, yields (3.9). \square

Theorem 3.6. For $(\bar{\alpha}_r)^n = (\alpha_0^n, \alpha_1^n, \dots, \alpha_{r-1}^n)$ and $(\bar{\alpha}_r)^m = (\alpha_0^m, \alpha_1^m, \dots, \alpha_{r-1}^m)$ we have

$$\begin{aligned} n^\ell \prod_{i=1}^r \sum_{s_i=0}^{n-1} (\alpha_{i-1})^{ms_i} M_\ell^{[0,r]} \left(\frac{x}{n} + \frac{\left(\sum_{i=1}^r s_i\right)m}{n}; k; 1, e, e; (\bar{\alpha}_r)^n \right) &= \\ m^{-rk+\ell} n^{rk} \prod_{i=1}^r \sum_{p_i=0}^{m-1} (\alpha_{i-1})^{np_i} M_\ell^{[0,r]} \left(\frac{x}{m} + \frac{\left(\sum_{i=1}^r p_i\right)n}{m}; k; 1, e, e; (\bar{\alpha}_r)^m \right). \quad (3.10) \end{aligned}$$

$$\begin{aligned} n^{\ell+r} \prod_{i=1}^r \sum_{s_i=0}^{n-1} (\alpha_{i-1})^{ms_i} M_{\ell+r}^{[0,r]} \left(\frac{x}{n} + \frac{\left(\sum_{i=1}^r s_i\right)m}{n}; k; 1, e, e; (\bar{\alpha}_r)^n \right) &= \\ \frac{m^{-r(k-1)+\ell} n^{rk}}{2^r} \prod_{i=1}^r \sum_{p_i=0}^{m-1} (\alpha_{i-1})^{np_i} M_\ell^{[0,r]} \left(\frac{x}{m} + \frac{\left(\sum_{i=1}^r p_i\right)n}{m}; k-1; 1, e, e; (\bar{\alpha}_r)^m \right). \quad (3.11) \end{aligned}$$

Proof. For the first equation and starting with (2.1), we get

$$\begin{aligned} \sum_{\ell=0}^{\infty} \frac{(nt)^\ell}{\ell!} \prod_{i=1}^r \sum_{s_i=0}^{n-1} (\alpha_{i-1})^{ms_i} M_\ell^{[0,r]} \left(\frac{x}{n} + \frac{\left(\sum_{i=1}^r s_i\right)m}{n}; k; 1, e, e; (\bar{\alpha}_r)^n \right) &= \frac{n^{rk} 2^{r(1-k)} t^{rk} e^{xt}}{\prod_{i=0}^{r-1} (\alpha_i^n e^{nt} - 1)} \frac{\prod_{i=0}^{r-1} (\alpha_i e^{nmt} - 1)}{\prod_{i=0}^{r-1} (\alpha_i^m e^{mt} - 1)} \\ &= \frac{n^{-rk} n^{rk} 2^{r(1-k)} t^{rk} m^{rk} e^{mt}}{\prod_{i=0}^{r-1} (\alpha_i^m e^{mt} - 1)} \prod_{i=1}^r \sum_{p_i=0}^{m-1} (\alpha_{i-1})^{np_i} \end{aligned}$$

$$= m^{-rk} n^{rk} \sum_{\ell=0}^{\infty} \left(\prod_{i=1}^r \sum_{p_i=0}^{m-1} (\alpha_{i-1})^{np_i} m^{\ell} M_{\ell}^{[0,r]} \left(\frac{x}{m} + \frac{\left(\sum_{i=1}^r p_i \right) n}{m}; k; 1, e, e; (\overline{\alpha}_r)^m \right) \right) \frac{t^{\ell}}{\ell!}.$$

Equating coefficients of t^{ℓ} on both sides, yields (3.10).

Also, It is not difficult to prove (3.11). \square

4. Some relations between the polynomials $M_n^{[m-1,r]}(x; k; a, b, c; \overline{\alpha}_r)$ and other polynomials and numbers

In this section, we give some relationships between the polynomials $M_n^{[m-1,r]}(x; k; a, b, c; \overline{\alpha}_r)$ and Laguerre polynomials, Jacobi polynomials, Hermite polynomials, generalized Stirling numbers of second kind, Stirling numbers and Bleimann-Butzer-hahn basic.

Theorem 4.1. For $\overline{\alpha}_r = (\alpha_0, \alpha_1, \dots, \alpha_r) \in \mathbb{C}$, $(x; \overline{\alpha})_{\underline{\ell}} = (x - \alpha_0)(x - \alpha_1) \dots (x - \alpha_{\ell-1})$ and $n, j \in \mathbb{N}_0$, we have relationship

$$M_n^{[m-1,r]}(x; k; a, b, c; \overline{\alpha}_r) = \sum_{j=0}^n (x; \alpha)_{\underline{j}} \sum_{\ell=j}^n \binom{n}{n-\ell} (\ln c)^{\ell} S(\ell, j; \overline{\alpha}) M_{n-\ell}^{[m-1,r]}(k; a, b, c; \overline{\alpha}_r) \quad (4.1)$$

between the new unification of Apostol-type polynomials and generalized Stirling numbers of second kind, see [4].

Proof. Using (3.4) and from definition of generalized Stirling numbers of second kind, we easily obtain (4.1). \square

Theorem 4.2. For $\overline{\alpha}_r = (\alpha_0, \alpha_1, \dots, \alpha_r) \in \mathbb{C}$, $(x)_{\underline{\ell}} = (x)(x-1) \dots (x-\ell+1)$ and $n, j \in \mathbb{N}_0$, we have relationship

$$M_n^{[m-1,r]}(x; k; a, b, c; \overline{\alpha}_r) = \sum_{j=0}^n (x)_{\underline{j}} \sum_{\ell=j}^n \binom{n}{n-\ell} (\ln c)^{\ell} S(\ell, j) M_{n-\ell}^{[m-1,r]}(k; a, b, c; \overline{\alpha}_r) \quad (4.2)$$

between the new unification of Apostol-type polynomials and Stirling numbers of second kind.

Proof. Using (3.4) and from definition of Stirling numbers of second kind, see [8], we easily obtain (4.2). \square

Theorem 4.3. The relationship

$$M_n^{[m-1,r]}(x; k; a, b, c; \overline{\alpha}_r) = \sum_{j=0}^n \sum_{\ell=j}^n (-1)^j \ell! \binom{n}{n-\ell} (\ln c)^{\ell} \binom{\ell + \alpha}{\ell - j} L_j^{(\alpha)}(x) M_{n-\ell}^{[m-1,r]}(k; a, b, c; \overline{\alpha}_r) \quad (4.3)$$

holds between the new unification of multiparameter Apostol-type polynomials and generalized Laguerre polynomials, see [26, No.(3) **Table1**].

Proof. From (3.4) and substitute

$$x^\ell = \ell! \sum_{j=0}^{\ell} (-1)^j \binom{\ell + \alpha}{\ell - j} L_j^\alpha(x),$$

then we get (4.3). \square

Theorem 4.4. For $(\alpha + \beta + j + 1)_{\ell+1} = (\alpha + \beta + j + 1)(\alpha + \beta + j + 2) \dots (\alpha + \beta + j + \ell + 1)$. The relationship

$$M_n^{[m-1, r]}(x; k; a, b, c; \overline{\alpha}_r) = \sum_{j=0}^n \sum_{\ell=j}^n (-1)^j \ell! \binom{n}{n-\ell} (\ln c)^\ell \binom{\ell + \alpha}{\ell - j} \frac{\alpha + \beta + 2j + 1}{(\alpha + \beta + j + 1)_{\ell+1}} P_j^{(\alpha, \beta)}(1 - 2x) M_{n-\ell}^{[m-1, r]}(k; a, b, c; \overline{\alpha}_r) \quad (4.4)$$

holds between the new unification of Apostol-type polynomials and Jacobi polynomials, see [22, p.49, Eq. (35)].

Proof. From (3.4) and substitute

$$x^\ell = \ell! \sum_{j=0}^{\ell} (-1)^j \binom{\ell + \alpha}{\ell - j} \frac{\alpha + \beta + 2j + 1}{(\alpha + \beta + j + 1)_{\ell+1}} P_j^{(\alpha, \beta)}(1 - 2x),$$

then we get (4.4). \square

Theorem 4.5. The relationship

$$M_n^{[m-1, r]}(x; k; a, b, c; \overline{\alpha}_r) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\ell=2j}^n 2^{-\ell} \binom{n}{n-\ell} \binom{\ell}{2j} \frac{2j!}{j!} (\ln c)^\ell H_{\ell-2j}(x) M_{n-\ell}^{[m-1, r]}(k; a, b, c; \overline{\alpha}_r) \quad (4.5)$$

holds between the new unification of Apostol-type polynomials and Hermite polynomials, see [26, No.(1) **Table1**].

Proof. From (3.4) and substitute

$$x^\ell = 2^{-\ell} \sum_{j=0}^{\lfloor \frac{\ell}{2} \rfloor} \binom{\ell}{2j} \frac{2j!}{j!} H_{\ell-2j}(x),$$

then we get (4.5). \square

Theorem 4.6. When $m = 1$, $a = 1$, $b = e$ and $c = e$ in (2.1) and for $\overline{\alpha}_r = (\alpha_0, \alpha_1, \dots, \alpha_{r-1})$, $\overline{\alpha}_r^* = \left(\frac{1}{\alpha_0}, \frac{1}{\alpha_1}, \dots, \frac{1}{\alpha_{r-1}}\right)$, $\alpha_i \neq 0$, $i = 0, 1, \dots, r-1$ and $\overline{\beta}_m = (\beta_0, \beta_1, \dots, \beta_{m-1})$, $\overline{\beta}_m^* = \left(\frac{1}{\beta_0}, \frac{1}{\beta_1}, \dots, \frac{1}{\beta_{m-1}}\right)$, $\beta_i \neq 0$, $i = 0, 1, \dots, m-1$, we have the following relationship

$$M_n^{(r)}(x; k; \overline{\alpha}_r) = \frac{n!}{\prod_{i=0}^{r-1} \alpha_i} \sum_{m=r}^{\infty} \frac{2^{(1-k)(r-m)} \prod_{j=0}^{m-1} \beta_j}{(n + k(m-1))!} C(m, r; \overline{\alpha}_r^*; \overline{\beta}_m^*) M_{n+k(m-r)}^{(m)}(x; k; \overline{\beta}_m), \quad (4.6)$$

between the new unified family of generalized Apostol-Euler, Bernoulli and Genocchi polynomials, and $C(m, r; \overline{\alpha}_r^*; \overline{\beta}_m^*)$ (the generalized Lah numbers), see [3].

Proof. From [7, Eq. 2.1],

$$\begin{aligned}
\sum_{n=0}^{\infty} M_n^{(r)}(x; k; \bar{\alpha}_r) \frac{t^n}{n!} &= \frac{t^{rk} 2^{r(1-k)} e^{xt}}{\prod_{i=1}^{r-1} (\alpha_i e^t - 1)} \\
&= \frac{t^{rk} 2^{r(1-k)} e^{xt}}{\prod_{i=1}^{r-1} \alpha_i} \frac{1}{(e^t; \bar{\alpha}^*)_r} \\
&= \frac{t^{rk} 2^{r(1-k)} e^{xt}}{\prod_{i=1}^{r-1} \alpha_i} \sum_{m=r}^{\infty} C(m, r; \bar{\alpha}^*_r; \bar{\beta}^*_m) \frac{1}{(e^t; \bar{\beta}^*)_m} \\
&= \sum_{m=r}^{\infty} \frac{t^{k(r-m)} 2^{(r-m)(1-k)} \prod_{j=1}^{m-1} \beta_j}{\prod_{i=1}^{r-1} \alpha_i} C(m, r; \bar{\alpha}^*_r; \bar{\beta}^*_m) \frac{t^{mk} 2^{m(1-k)} e^{xt}}{(e^t; \bar{\beta}^*)_m} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=r}^{\infty} \frac{n! 2^{(1-k)(r-m)} \prod_{j=0}^{m-1} \beta_j}{(n+k(m-1))! \prod_{i=0}^{r-1} \alpha_i} C(m, r; \bar{\alpha}^*_r; \bar{\beta}^*_m) M_{n+k(m-r)}^{(m)}(x; k; \bar{\beta}_m) \right) \frac{t^n}{n!}.
\end{aligned}$$

Equating the coefficients of t^n on both sides, yields (4.6). \square

Using **No.13** in **Table 1**, see [7] and the definition of the unified Bernstein and Bleimann-Butzer-Hahn basis(see [18]),

$$\left(\frac{2^{1-k} x^k t^k}{(1+ax)^k} \right)^m \frac{1}{mk!} e^{t(\frac{1+bx}{1+ax})} = \sum_{n=0}^{\infty} p_n^{(a,b)}(x; k, m) \frac{t^n}{n!}, \quad (4.7)$$

where $k, m \in \mathbb{Z}^+$, $a, b \in \mathbb{R}$, $t \in \mathbb{C}$, we obtain the following theorem

Theorem 4.7. For $\alpha_i \neq 0, i = 0, 1, \dots, r-1$, we have relationship

$$P_n^{(a,b)}(x; k, r) = \frac{\prod_{i=0}^{r-1} \alpha_i}{rk!} \left(\frac{x}{1+ax} \right)^{rk} \sum_{j=0}^r s \left(r, j; \frac{1}{\bar{\alpha}_r} \right) \sum_{\ell=0}^n j^{n-\ell} \binom{n}{\ell} M_{\ell}^{(r)} \left(\frac{1+bx}{1+ax}; k; \bar{\alpha}_r \right) \quad (4.8)$$

between the unified Bernstein and Bleimann-Butzer-Hahn basis, the new unified family of generalized Apostol-Bernoulli, Euler and Genocchi polynomials, see [7] and generalized Stirling numbers of first kind, see [4].

Proof. From (2.1) and (4.7) and with some elementary calculation, we easily obtain (4.8). \square

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